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INTEGRAL GEOMETRY OF EQUIDISTANTS IN HYPERBOLIC SPACE

BY

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ABSTRACT

We generalize the classical formulas of integral geometry, by getting integral geometric formulas for the intersection of a fixed compact hypersurface of hyperbolic space and a moving totally umbilical hypersurface. In particular we compute the mean value of the volume, the total mean curvatures and the Euler characteristic of these intersections when the totally umbilical hypersurface moves over all the intersecting positions. Analogous formulas are given for totally umbilical hypersurfaces contained in totally geodesic planes of $\mathbb{H}^n$.

1. Introduction

Consider the space $\mathcal{L}_r$ of r-dimensional affine subspaces (planes) in n-dimensional *euclidean* space. Let $\mathrm{d}L_r$ be a measure in $\mathcal{L}_r$ invariant under the action of rigid motions. A classical result in integral geometry states that the integral of the Euler characteristic of the intersections of r-dimensional planes with a compact domain Q with smooth boundary $S = \partial Q$ is

$$\int_{\mathcal{L}_r} \chi(L_r \cap Q)\mathrm{d}L_r = c\int_S \sigma_{r-1}(x)\mathrm{d}x$$

where c is a constant, σ_i stands for the i-th mean curvature of S and $\mathrm{d}x$ is the volume element of S. If Q is convex, this gives the measure of planes intersecting it.

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In space forms, analogous formulas were obtained in [San76]. More precisely, let $\mathcal{L}_r$ be the space of r-dimensional totally geodesic submanifolds of $M^n(k)$, the n-dimensional simply connected manifold of curvature k. Consider a measure $\mathrm{d}L_r$ in $\mathcal{L}_r$ invariant under the action of isometries of $M^n(k)$. For a compact domain $Q \subset \mathbb{S}^n(k)$ with smooth boundary $S = \partial Q$,

$$\int_{\mathcal{L}_r} \chi(L_r \cap Q)\mathrm{d}L_r = c\,\mathrm{vol}(Q) + \sum_{2i<r} c_i\ k^{[r/2]-i} \int_S \sigma_{2i}(x)\mathrm{d}x$$

where the constants depend only on the dimensions and vol stands for the volume.

To go one step further in generalizing these formulas, it is natural to substitute totally geodesic hypersurfaces by totally umbilic hypersurfaces. In euclidean and spherical spaces, totally umbilic submanifolds are spheres so one can reduce to a particular case of kinematic formulas (which measure sets of congruent figures intersecting a given one).

Nevertheless, in *hyperbolic* geometry there are many open totally umbilic hypersurfaces apart from geodesic hyperplanes. It is well known (cf. [doCar]) that totally umbilical hypersurfaces have constant normal curvature λ and they can be classified into 4 types: geodesic spheres ($\lambda < 1$), horospheres ($\lambda = 1$), equidistants ($0 < \lambda < 1$), and totally geodesic hyperplanes ($\lambda = 0$). **Geodesic spheres** are sets of points at given distance from a center. **Horospheres** are obtained by making the center of a sphere go to infinity. **Equidistants** are connected components of tubes about totally geodesic hyperplanes.

This paper generalizes the described integral geometric formulas to totally umbilical hypersurfaces of hyperbolic space. The role of r-dimensional totally geodesic submanifolds will be played by umbilical hypersurfaces of $(r+1)$-dimensional totally geodesic submanifolds. For spheres and horospheres, these formulas were already obtained by E. Gallego, A. M. Naveira, and the author in [GNS]. Here we find new results for the lower values of the normal curvature λ and for higher codimensions.

2. Definitions and invariant measures

The n-dimensional hyperbolic space is any complete simply connected riemannian manifold with constant curvature -1. We will use the hyperboloid model; let $(\mathbb{R}^{n+1}, \langle,\rangle)$ be the Minkowski space with

$$\langle x, y\rangle = -x_0y_0 + x_1y_1 + \cdots + x_ny_n.$$

Hyperbolic space is the upper connected component of the two-sheet hyperboloid defined by $\langle,\rangle$, namely

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} | \langle x, x\rangle = -1,\ x_0 > 0\},$$

with the riemannian metric defined by the restriction of $\langle,\rangle$. The Levi-Civita connection ∇ in $\mathbb{H}^n$ is the orthogonal projection, with respect to $\langle,\rangle$, onto $T\mathbb{H}^n$ of the usual connection in $\mathbb{R}^{n+1}$. As a consequence of this, geodesics in this model appear as intersections with $\mathbb{H}^n$ of 2-dimensional linear subspaces.

Let G be the group of isometries of $\mathbb{H}^n$. It is known that G is the group of linear endomorphisms of $\mathbb{R}^{n+1}$ preserving $\langle,\rangle$ and $\mathbb{H}^n$. That is,

$$G = \{g \in Gl(n+1, \mathbb{R}) | g^t J g = J,\ g_0^0 > 0\}$$

where J is a diagonal matrix with -1 in the first position and 1 in the rest. We will think about the elements $g \in G$ as orthonormal frames; the first column g_0 is a point of $\mathbb{H}^n$ and the rest $(g_1, \ldots, g_n)$ form an orthonormal basis of $T_{g_0}\mathbb{H}^n$. For each i, consider $g_i \colon G \to \mathbb{R}^{n+1}$ mapping each matrix to its i-th column. The differential of this map is a 1-form with values in $\mathbb{R}^{n+1}$, and can be expressed as follows:

$$\mathrm{d}g_i = -\omega_i^0 g_0 + \omega_i^1 g_1 + \cdots + \omega_i^n g_n \tag{1}$$

for the invariant 1-forms

$$\omega_i^j(X) = \langle g_i, X^j\rangle, \quad X \in T_g G$$

where X^j denotes the j-th column of X. These differential forms verify

$$\omega_0^h = \omega_h^0, \quad \omega_i^j = -\omega_j^i, \quad 0 < h, i, j \leq n.$$

Taking exterior derivative in (1) gives the structure equations

$$\mathrm{d}\omega_i^j = -\omega_i^0 \wedge \omega_0^j + \sum_{h \neq i,j} \omega_i^h \wedge \omega_h^j, \quad 0 \leq i < j \leq n. \tag{2}$$

Recall that a point in a hypersurface is called **umbilical** when the normal curvatures in all directions of this point are equal. A hypersurface such that all the points are umbilical is called **totally umbilical**. In constant curvature ambients, the normal curvatures of totally umbilical hypersurfaces are the same at all the points (cf. [doCar]).

Definition 2.1: For $\lambda \geq 0$, a complete totally umbilical hypersurface with normal curvature λ will be called a **λ-geodesic hyperplane** of $\mathbb{H}^n$. Denote $\mathcal{L}_{n-1}^{\lambda}$ the set of all λ-geodesic hyperplanes of $\mathbb{H}^n$.

This definition includes, for $\lambda = 1$, horospheres, and for $\lambda = 0$, geodesic hyperplanes. The case $\lambda > 1$ consists of spheres of radius $\operatorname{arctanh}(1/\lambda)$. For $\lambda < 1$, the tube at distance $\operatorname{arctanh}\lambda$ around a geodesic hyperplane has two connected components, each being a λ-geodesic hyperplane. It is not difficult to see that in the hyperboloid model, λ-geodesic hyperplanes are intersections of $\mathbb{H}^n$ with affine hyperplanes of the type

$$\{x \in \mathbb{R}^{n+1} \,|\langle x, y\rangle = -\lambda\}$$

where $\langle y, y\rangle = 1$.

Now we introduce a higher codimension analog of λ-geodesic hyperplanes. Since a geodesic r-plane $L_r \subset \mathbb{H}^n$ is isometric to $\mathbb{H}^r$, it makes sense to talk about λ-geodesic hyperplanes of L_r.

Definition 2.2: A λ-geodesic hyperplane of some geodesic $(r+1)$-plane in $\mathbb{H}^n$ will be called a **λ-geodesic r-plane**. Define $\mathcal{L}_r^{\lambda}$ to be the set of all such λ-geodesic r-planes. ∎

Remark: Using the Gauss equation one immediately gets that the λ-geodesic r-planes are riemannian manifolds of constant sectional curvature $\lambda^2 - 1$ (for $r > 1$). Since they are simply connected, the Cartan theorem assures that they are isometric to spheres ($\lambda > 1$), euclidean space ($\lambda = 1$) or hyperbolic spaces ($0 \leq \lambda < 1$).

Let $e_0, \ldots, e_n$ be the canonical reference of $\mathbb{R}^{n+1}$ and consider the $(r+1)$-dimensional geodesic plane $L_{r+1} = \langle e_0, \ldots, e_{r+1}\rangle \cap \mathbb{H}^n$. Fix L_r^{λ} the λ-geodesic r-plane through e_0, contained in L_{r+1} and such that e_{r+1} is normal at e_0 and points towards the convexity of L_r^{λ}. Let H_r be the subgroup of isometries leaving L_r^{λ} invariant. Now, $\mathcal{L}_r^{\lambda}$ is identified with the homogeneous space G/H_r. This defines a canonical projection $\pi_r\colon G \to L_r^{\lambda}$. The following proposition allows us to find the (unique up to a constant factor) invariant measure in a homogeneous space.

PROPOSITION 2.1: *Let G/H be a homogeneous space of dimension m. Let*

$$\eta = \theta^1 \wedge \cdots \wedge \theta^m$$

where θ^i are left invariant 1-forms defining the foliation gH ($g \in G$). This form η is closed if and only if there exists a measure α in G/H which is invariant by the action of G. In this case $\eta = \pi^*\alpha$ for the canonical projection $\pi\colon G \to G/H$.

PROPOSITION 2.2: *The space $\mathcal{L}_r^\lambda$ admits a measure $\mathrm{d}L_r^\lambda$ invariant with respect to isometries which is defined by*

$$\pi_r^*(\mathrm{d}L_r^\lambda) = \bigwedge_{h=1}^{r} (\omega_h^{r+1} - \lambda\omega_0^h) \wedge \Big(\bigwedge_{1\le i\le r+1<j\le n} \omega_i^j \Big) \wedge \bigwedge_{k=r+1}^{n} \omega_0^k.$$

Proof: Structure equations (2) show that the differential form is closed. It remains to prove that $\omega_h^n - \lambda\omega_0^h$, ω_i^j and ω_0^k are null on H. Take a curve $g(t) = (g_0(t), \dots, g_n(t))$ in H. Note g_i and $\dot g_i$ the position and tangent vectors of $g_i(t)$ at $t = 0$. Clearly, $\dot g_0$ is tangent to L_r^λ so

$$\omega_0^k(\dot g) = \omega_k^0(\dot g) = \langle g_k, \dot g_0\rangle = 0.$$

Considering L_r^λ as a hypersurface of $L_{r+1} \cong \mathbb{H}^r$, $g_{r+1}(t)$ is the unit normal vector and

$$\omega_h^{r+1}(\dot g) = \langle g_h, \dot g_{r+1}\rangle = \langle g_h, \nabla_{\dot g_0} g_{r+1}\rangle = \langle g_h, \lambda \dot g_0\rangle$$

since all the directions on L_r^λ are principal with normal curvature λ. Thus, $(\omega_h^{r+1} - \lambda\omega_0^h)(\dot g)$ is null because

$$\omega_0^h(\dot g) = \omega_h^0(\dot g) = \langle g_h, \dot g_0\rangle,$$

Finally, since $g_j \in \langle e_{r+2}, \dots, e_n\rangle$,

$$\omega_i^j(\dot g) = \langle g_i, \dot g_j\rangle = 0 \quad \text{for } 1 \le i \le r+1 < j \le n. \qquad \blacksquare$$

Remark: Note that

$$\bigwedge_{h=1}^{r} (\omega_h^{r+1} - \lambda\omega_0^h) \wedge \omega_0^{r+1}$$

corresponds to the measure $\mathrm{d}L_{[r+1]r}^\lambda$ of λ-geodesic hyperplanes of an $(r+1)$-dimensional geodesic plane L_{r+1}. On the other hand,

$$\Big(\bigwedge_{1\le i\le r+1<j\le n} \omega_i^j \Big) \wedge \bigwedge_{k=r+2}^{n} \omega_0^k$$

corresponds to the measure of $(r+1)$-dimensional geodesic planes in $\mathbb{H}^n$ (cf. [San76]). Therefore, abusing the notation, we will write

$$\mathrm{d}L_r^\lambda = \mathrm{d}L_{[r+1]r}^\lambda \wedge \mathrm{d}L_{r+1}. \tag{3}$$

We have paid no attention to the signs because we are not dealing with differential forms but with measures.

Given polar coordinates $(\rho, u) \in \mathbb{R} \times \mathbb{S}^{n-1}$, one can consider the geodesic γ going by the origin with tangent u, and the λ-geodesic hyperplane such that $\dot\gamma(\rho)$ is the exterior normal at the intersection. With these coordinates it can be seen that

$$\mathrm{d}L_r^\lambda = (\cosh\rho - \lambda\sinh\rho)^{n-1}\mathrm{d}\rho\mathrm{d}u$$

where $\mathrm{d}u$ stands for the volume element of $\mathbb{S}^{n-1}$.

3. Volume of intersections with λ-geodesic planes

The next proposition generalizes to λ-geodesic planes the formula for the mean volume of intersection with geodesic planes ([San76], p. 245). Here and in the following, O_i will denote the volume of the unit i-dimensional sphere.

PROPOSITION 3.1: *Let S^q be a q-dimensional compact submanifold in $\mathbb{H}^n$, piecewise C^1, possibly with boundary. Then*

$$\int_{\mathcal{L}_{n-1}^\lambda} \mathrm{vol}_{q-1}(L_{n-1}^\lambda \cap S^q)\mathrm{d}L_{n-1}^\lambda = \frac{O_n O_{q-1}}{O_q} \cdot \mathrm{vol}_q(S)$$

where vol_i denotes the i-dimensional volume.

Proof: Consider the manifold

$$E(S) = \{(L_{n-1}^\lambda, p) \in \mathcal{L}_{n-1}^\lambda \times S \,|\, p \in L_{n-1}^\lambda \cap S\}.$$

For almost all (L_{n-1}^λ, p), i.e. out of a zero measure subset of $E(S)$, the intersection $L_{n-1}^\lambda \cap S$ is a C^1 submanifold in a neighborhood of p. Denote $\mathrm{d}x_{q-1}$ the volume element of this submanifold. Now,

$$\int_{\mathcal{L}_{n-1}^\lambda} \mathrm{vol}_{q-1}(L_{n-1}^\lambda \cap S)\mathrm{d}L_r^\lambda = \int_{E(S)} \mathrm{d}x_{q-1} \wedge \mathrm{d}L_r^\lambda$$

where $\mathrm{d}L_r^\lambda$ denotes the volume element on $\mathcal{L}_{n-1}^\lambda$ and also its pull-back to $E(S)$. Consider now

$$G(S) = \{g \in G | g_0 \in S \quad g_1, \dots, g_{q-1} \in T_{g_0}S \quad g_{q+1}, \dots, g_{n-1} \perp T_{g_0}S\}$$

and the projection $\pi\colon G(S) \to E(S)$ which maps the reference g to the λ-geodesic hyperplane through g_0, tangent to $g_1, \dots, g_{n-1}$ and with the convexity in the side of g_n. This way,

$$\begin{aligned}\pi^*(\mathrm{d}x_{q-1} \wedge \mathrm{d}L^{\lambda}_{n-1}) &= \bigwedge_{h=1}^{q-1} \omega_0^h \wedge \omega_0^n \wedge \bigwedge_{i=1}^{n-1} (\omega_i^n - \lambda\omega_0^i) \\ &= \bigwedge_{h=1}^{q-1} \omega_0^h \wedge \omega_0^n \wedge \bigwedge_{i=1}^{q-1} \omega_i^n \wedge \bigwedge_{i=q}^{n-1} (\omega_i^n - \lambda\omega_0^i).\end{aligned}$$

Given $g \in G(S)$, take $\overline{g} \in G(S)$ such that

$$\overline{g}_0 = g_0, \dots, \overline{g}_{q-1} = g_{q-1} \quad \text{and} \quad \overline{g}_q \in T_{g_0}S.$$

For every $v \in T_gG(S)$,

$$\begin{aligned}\omega_0^i(v) &= \langle g_i, \mathrm{d}g_0(v)\rangle = 0, \quad q < i < n, \\ \overline{\omega}_0^i(v) &= \langle \overline{g}_i, \mathrm{d}g_0\rangle = 0, \quad q < i.\end{aligned}$$

Therefore,

$$\omega_0^n = \sum_{i=q}^{n} \langle \overline{g}_i, g_n\rangle \overline{\omega}_0^i = \langle \overline{g}_q, g_n\rangle \overline{\omega}_0^q \quad \text{and} \quad \omega_0^q = \sum_{i=q}^{n} \langle \overline{g}_i, g_q\rangle \overline{\omega}_0^i = \langle \overline{g}_q, g_q\rangle \overline{\omega}_0^q.$$

Since we are working with measures, no attention must be paid to the sign changes and we can write

$$\text{(4)} \quad \pi^*(\mathrm{d}x_{q-1} \wedge \mathrm{d}L^{\lambda}_{n-1}) = \langle \overline{g}_q, g_n\rangle \bigwedge_{h=1}^{q-1} \omega_0^h \wedge \overline{\omega}_0^q \wedge \bigwedge_{i=1}^{n-1} \omega_i^n = |\sin\theta| \mathrm{d}\mathbb{S}^{n-1} \wedge \mathrm{d}x_q$$

where $\mathrm{d}\mathbb{S}^{n-1}$ is the volume element in $\mathbb{S}^{n-1}$ corresponding to the normal vector of L^{λ}_{n-1} in x, $\mathrm{d}x_q$ corresponds to the volume element of S and θ is the angle between S and L^{λ}_{n-1} in x. Integrating both sides of (4) we get

$$\int_{E(S)} \mathrm{d}x_{q-1} \wedge \mathrm{d}L^{\lambda}_{n-1} = \int_{G(S)} \pi^* \mathrm{d}x_{q-1} \wedge \mathrm{d}L^{\lambda}_{n-1} = \int_{\mathbb{S}^{n-1}} |\sin\theta| \mathrm{d}\mathbb{S}^{n-1} \cdot \int_S \mathrm{d}x_q.$$

It is not difficult to compute that

$$\int_{\mathbb{S}^{n-1}} |\sin\theta| \mathrm{d}\mathbb{S}^{n-1} = \frac{O_n O_{q-1}}{O_q}. \qquad \blacksquare$$

Consider the case of λ-geodesic planes with higher codimension.

PROPOSITION 3.2: *Let S^q be a q-dimensional compact submanifold in $\mathbb{H}^n$, piecewise C^1, possibly with boundary. Then if $r+q \geq n$,*

$$\int_{\mathcal{L}_r^\lambda} \mathrm{vol}_{r+q-n}(L_r^\lambda \cap S^q)\mathrm{d}L_r^\lambda = \frac{O_n \cdots O_{n-r-1} O_{r+q-n}}{O_r \cdots O_0 O_q} \cdot \mathrm{vol}_q(S).$$

Proof: Using (3) and the last proposition,

$$\begin{aligned}\int_{\mathcal{L}_r^\lambda} \mathrm{vol}_{r+q-n}(L_r^\lambda \cap S)\mathrm{d}L_r^\lambda &= \int_{\mathcal{L}_{r+1}} \int_{\mathcal{L}_{[r+1]r}^\lambda} \mathrm{vol}_{r+q-n}(L_r^\lambda \cap S)\mathrm{d}L_{[r+1]r}^\lambda \mathrm{d}L_{r+1} \\ &= \frac{O_{r+1}O_{r+q-n}}{O_{r+1+q-n}} \int_{\mathcal{L}_{r+1}} \mathrm{vol}_{r+1+q-n}(S \cap L_{r+1})\mathrm{d}L_{r+1}.\end{aligned}$$

The formula for the integral of the volume of intersections with geodesic planes (cf. [San76], p. 245) gives

$$\int_{\mathcal{L}_{r+1}} \mathrm{vol}_{r+1+q-n}(S \cap L_{r+1})\mathrm{d}L_{r+1} = \frac{O_n \cdots O_{n-r-1} O_{r+1+q-n}}{O_{r+1} \cdots O_1 O_0 O_q} \mathrm{vol}_q(S). \qquad \blacksquare$$

Note that for $\lambda = 0$, these results coincide with those in ([San76]) except for the constant O_{n-r-1}. This is coherent with the fact that, even for $\lambda = 0$, the space $\mathcal{L}_r^\lambda$ is a fiber bundle of base $\mathcal{L}_r$ and fiber $\mathbb{S}^{n-r-1}$.

For $r+q=n$, we have an analog of the Cauchy–Crofton formula

$$\int_{\mathcal{L}_r^\lambda} \#(L_r^\lambda \cap S^q)\mathrm{d}L_r^\lambda = \frac{O_n \cdots O_{n-r+1} O_{n-r-1}}{O_r \cdots O_1} \cdot \mathrm{vol}_q(S).$$

In particular, the integral of the number of intersection points of λ-geodesic hyperplanes with a curve of length L is $4L/(O_{n-1} \cdots O_2)$. When $\lambda = 1$, this coincides with a result by Santaló for the cases $n = 2, 3$ (cf. [San67], [San68]) and by Gallego, Naveira and the author for general n (cf. [GNS]).

4. Total mean curvatures of intersections with λ-geodesic planes

Apart from the volume, the most natural integral geometric invariants associated to hypersurfaces are total mean curvatures

$$M_i(S) := \int_S \sigma_i(x)\mathrm{d}x$$

where $\mathrm{d}x$ is the volume element of S and σ_i is the i-th mean curvature of S at x. Here we deduce reproductive formulas for these invariants analogous to those given for the volume.

Let $L = L^{\lambda}_{n-1}$ be a λ-geodesic hyperplane intersecting transversally a hypersurface S in x. This defines, at least locally, a submanifold $C = L \cap S$ of codimension 2. The second fundamental forms of these submanifolds are bilinear symmetric forms given by

$$h_L\colon (T_xL) \times (T_xL) \to (T_xL)^{\perp}, \quad \nabla_X Y = \nabla^L_X Y + h_L(X,Y);$$
$$h_S\colon (T_xS) \times (T_xS) \to (T_xS)^{\perp}, \quad \nabla_X Y = \nabla^S_X Y + h_S(X,Y);$$
$$h_C\colon (T_xC) \times (T_xC) \to (T_xC)^{\perp}, \quad \nabla_X Y = \nabla^C_X Y + h_C(X,Y).$$

Here ∇^M denotes the connection on the submanifold M. One can also consider the second fundamental form h^L_C of C as a submanifold of L. Clearly,

$$h_C(X,Y) = h^L_C(X,Y) + h_L(X,Y) = h^S_C(X,Y) + h_S(X,Y). \tag{5}$$

Let N_S and N_L be the inner unit normal vectors of S and L in a point $x \in C = S \cap L$. For $X, Y \in T_xC$ one has

$$h_S(X,Y) = \mu_S(X,Y) \cdot N_S, \quad h_L(X,Y) = \mu_L(X,Y) \cdot N_L,$$

where μ_S and μ_L are real-valued bilinear forms on T_xS and T_xL, respectively. On the other hand, for some real-valued bilinear form μ^L_C on T_xC,

$$h^L_C(X,Y) = \mu^L_C(X,Y) N_C$$

where $N_C \in T_xL$ is the inner unit normal vector of C.

PROPOSITION 4.1: *With the above notations,*

$$\mu_S = \cos\theta \mu_L + \sin\theta \mu^L_C$$

where θ is the angle between N_L and N_S.

Proof: Using (5),

$$\begin{aligned}\mu_S(X,Y) &= \langle h_S(X,Y), N_S\rangle = \langle h_C(X,Y), N_S\rangle \\ &= \mu^L_C(X,Y)\langle N_C, N_S\rangle + \mu_L(X,Y)\langle N_L, N_S\rangle, \quad \blacksquare\end{aligned}$$

Since $\mu_L = \lambda Id$ we can express μ^L_C in terms of the restriction of μ_S to T_xC,

$$\mu^L_C = \frac{\mu_L}{\sin\theta} - \frac{\lambda Id}{\tan\theta}. \tag{6}$$

To avoid confusion, fix the following notation. Given a (real-valued) symmetric bilinear form μ in a space of dimension r, we denote

$$\sigma_j(\mu) = \frac{\{k_{i_1} \ldots k_{i_j}\}}{\binom{r}{j}}$$

where $\{k_{i_1}\cdots k_{i_j}\}$ is the j-th elementary symmetric polynomial of the eigenvalues $k_1,\ldots,k_r$ of μ. Recall that

$$\det(\mu+tId)=\sum_{j=0}^{r}\{k_{i_1}\cdots k_{i_j}\}t^{r-j}.$$

With this notation the *j-th mean curvature* σ_j^S of the hypersurface S is $\sigma_j(\mu_S)$.

PROPOSITION 4.2: *The mean curvatures $\sigma_k(\mu_C^L)$ of C as a hypersurface of L are given by*

$$\sigma_k(\mu_C^L)=\sum_{l=0}^{k}\frac{\binom{n-l-2}{n-k-2}\binom{n-2}{l}}{\binom{n-2}{k}}(-1)^{k-l}\frac{\cos^{k-l}\theta}{\sin^k\theta}\lambda^{k-l}\sigma_l(\mu_P^S)$$

where μ_P^S is the restriction of μ_S to $P=T_xC$.

Proof:

$$\mu_C^L+tId=\frac{\mu_P^S}{\sin\theta}+\Big(t-\frac{\lambda}{\tan\theta}\Big)Id=\frac{1}{\sin\theta}(\mu_P^S+(t\sin\theta-\lambda\cos\theta)Id).$$

Taking determinants

$$\sum_{j=0}^{n-2}\binom{n-2}{j}\sigma_{n-j-2}(\mu_C^L)t^j$$

$$=\det(\mu_C+tId)$$

$$=\frac{\det(\mu_P^S+(t\sin\theta-\lambda\cos\theta)Id)}{\sin^{n-2}\theta}$$

$$=\frac{1}{\sin^{n-2}\theta}\sum_{i=0}^{n-2}\binom{n-2}{i}\sigma_{n-i-2}(\mu_P^S)(t\sin\theta-\lambda\cos\theta)^i$$

$$=\frac{1}{\sin^{n-2}\theta}\sum_{i=0}^{n-2}\binom{n-2}{i}\sigma_{n-i-2}(\mu_P^S)\sum_{j=0}^{i}\binom{i}{j}(-1)^{i-j}\lambda^{i-j}\cos^{i-j}\theta\sin^j\theta t^j$$

$$=\sum_{j=0}^{n-2}\frac{1}{\sin^{n-j-2}\theta}\Big(\sum_{i=j}^{n-2}\binom{i}{j}\binom{n-2}{i}(-1)^{i-j}\lambda^{i-j}\cos^{i-j}\theta\sigma_{n-i-2}(\mu_P^S)\Big)t^j. \qquad ∎$$

The following lemma is a generalization of the fact that, for surfaces in $\mathbb{R}^3$, the mean curvature in a point of a surface is the mean value of the normal curvatures in all the directions by this point.

LEMMA 4.1: *Let S be a hypersurface of some n-dimensional riemannian manifold and let μ be the second fundamental form of S in a point x. For every i-dimensional linear subspace L of T_xS, denote $\mu|_L$ the restriction of μ to L. Then, for $j \leq i < n-1$,*

$$\int_{G(T_xS,i)} \sigma_j(\mu|_L)\mathrm{d}L = \mathrm{vol}(G(n-1,i))\sigma_j(\mu).$$

Note that, if another hypersurface N intersects S orthogonally in x in such a way that $T_xN \cap T_xS = L$, then $\mu|_L$ is the second fundamental form of $S \cap L$ in x as a hypersurface of N. This is immediate by Proposition 4.1.

Proof: For the case $j = i$ this is a well-known result (cf. [LaSh, Teu]). For $j < i$, we can reduce to the preceding case as follows:

$$\begin{aligned}
&\int_{G(T_xS,i)} \sigma_j(\mu|_L)\mathrm{d}L \\
&= \int_{G(T_xS,i)} \left(\mathrm{vol}(G(i,j))^{-1} \int_{G(T_x(S\cap P_L),j)} \sigma_j(\mu|_l)\mathrm{d}l\right)\mathrm{d}L \\
&= \mathrm{vol}(G(i,j))^{-1} \int_{G(T_xS,j))} \int_{G(l^\perp,i-j)} \sigma_j(\mu|_l)\mathrm{d}L\mathrm{d}l \\
&= \mathrm{vol}(G(i,j))^{-1}\,\mathrm{vol}(G(n-j-1,i-j)) \int_{G(T_xS,j))} \sigma_j(\mu|_l)\mathrm{d}l \\
&= \mathrm{vol}(G(i,j))^{-1}\,\mathrm{vol}(G(n-j-1,i-j))\,\mathrm{vol}(G(n-1,j))\sigma_j(\mu). \qquad \blacksquare
\end{aligned}$$

Given a hypersurface $S \subset \mathbb{H}^n$, for almost every λ-geodesic hyperplane L_{n-1}^λ, the intersection $L_{n-1}^\lambda \cap S$ is a smooth hypersurface of L_{n-1}^λ. In this case, it makes sense to consider $M_i(L_{n-1}^\lambda \cap S)$, the total mean curvatures of $L_{n-1}^\lambda \cap S$ as a hypersurface of L_{n-1}^λ.

PROPOSITION 4.3: *Let S be a hypersurface of $\mathbb{H}^n$,*

$$\int_{\mathcal{L}_{n-1}^\lambda} M_j(L_{n-1}^\lambda \cap S)\mathrm{d}L_{n-1}^\lambda = \sum_{l=0}^{[j/2]} c_{l,j}^n \lambda^{2l} M_{j-2l}(S),$$

where

$$c_{l,j}^n = \frac{\binom{n-j+2l-2}{n-j-2}\binom{n-2}{j-2l}}{\binom{n-2}{j}} \frac{O_{n-2}O_{n-j+2l}O_0}{O_{n-j-1}O_{2l}}.$$

Proof: Denote $C = L_{n-1}^\lambda \cap S$. Using (4)

$$\int_{\mathcal{L}_{n-1}^\lambda} \int_C \sigma_j^C \mathrm{d}x\mathrm{d}L_{n-1}^\lambda = \int_S \int_{\mathbb{S}^{n-1}} \sin\theta\sigma_j^C \mathrm{d}\mathbb{S}^{n-1}\mathrm{d}x.$$

By Proposition 4.2, if μ_P^S is the restriction of μ_S to $P = T_xC$,

$$\binom{n-2}{j}\int_{\mathbb{S}^{n-1}} \sin\theta\sigma_j^C \mathrm{d}\mathbb{S}^{n-1}$$

$$= \sum_{i=0}^{j}\binom{n-i-2}{n-j-2}\binom{n-2}{i}(-1)^{j-i}\lambda^{j-i}\int_{\mathbb{S}^{n-1}}\frac{\sin\theta}{\sin^j\theta}\cos^{j-i}\theta\sigma_i(\mu_P^S)\mathrm{d}\mathbb{S}^{n-1}.$$

Taking 'polar coordinates' in $\mathbb{S}^{n-1}$, the latter integrals are

$$\int_{\mathbb{S}^{n-2}}\int_0^{\pi}\sin\theta\frac{1}{\sin^j\theta}\cos^{j-i}\theta\sigma_i(\mu_P^S)\sin^{n-2}\theta\mathrm{d}\theta\mathrm{d}P$$

$$= \int_0^{\pi}\sin^{n-j-1}\theta\cos^{j-i}\theta\mathrm{d}\theta\int_{\mathbb{S}^{n-2}}\sigma_i(\mu_P^S)\mathrm{d}P.$$

Using Lemma 4.1 gives the sought formula. The constants are easily obtained. ∎

Corollary 4.1: *For $j \le r-1$,*

$$\int_{\mathcal{L}_r^\lambda} M_j(L_r^\lambda \cap S)\mathrm{d}L_r^\lambda = \sum_{l=0}^{[j/2]} c_{l,j,r}^n \lambda^{2l} M_{j-2l}(S)$$

where

$$c_{l,j,r}^n = \frac{\binom{r-j+2l-1}{r-j-1}\binom{r-1}{j-2l}}{\binom{r-1}{j}}\frac{O_{n-2}\cdots O_{n-r-1}O_{n-j+2l}}{O_{r-2}\cdots O_1 O_{r-j}O_{2l}}.$$

Remark: For $j=0$ we recover the case $q=n-1$ of Proposition 3.2.

Proof: Formula (3) gives

$$\int_{\mathcal{L}_r^\lambda} M_j(L_r^\lambda \cap S)\mathrm{d}L_r^\lambda = \int_{\mathcal{L}_{r+1}}\int_{\mathcal{L}_{[r+1]r}^\lambda} M_j(L_r^\lambda\cap S)\mathrm{d}L_r^\lambda \mathrm{d}L_{r+1}$$

which, by the last proposition, is equal to

$$\int_{\mathcal{L}_{r+1}}\left(\sum_{l=0}^{[j/2]} c_{l,j}^{r+1}\lambda^{2l}M_{j-2l}(S\cap L_{r+1})\right)\mathrm{d}L_{r+1}.$$

Finally, the reproducibility formulas of M_i for intersections with geodesic planes that appear in ([San76], p. 248) give

$$\int_{\mathcal{L}_{r+1}} M_{j-2l}(S\cap L_{r+1})\mathrm{d}L_{r+1} = \frac{O_{n-2}\cdots O_{n-r-1}O_{n-j+2l}}{O_{r-1}\cdots O_0 O_{r-j+2l+1}}M_{j-2l}(S).$$ ∎

5. Measure of λ-geodesic planes intersecting a domain

In the following we express the integral of the Euler characteristic of the intersection of λ-geodesic planes with a domain of $\mathbb{H}^n$ in terms of the total mean curvatures of its boundary.

THEOREM 1: *Let $Q \subset \mathbb{H}^n$ be a compact domain with smooth boundary. For r even*

$$\int_{\mathcal{L}_r^\lambda} \chi(Q \cap L_r^\lambda) \mathrm{d}L_r^\lambda = (\lambda^2-1)^{r/2} \frac{O_{n-1}\cdots O_{n-r-1}}{O_r \cdots O_1} \cdot V(Q)$$

$$+\sum_{j=1}^{r/2}\left(\sum_{i=j}^{r/2}\binom{r-1}{2i-1}\frac{2c^n_{i-j,2i-1,r}}{O_{2i-1}O_{r-2i}}(\lambda^2-1)^{\frac{r-2i}{2}}\lambda^{2i-2j}\right)M_{2j-1}(\partial Q).$$

For r odd

$$\int_{\mathcal{L}_r^\lambda} \chi(Q \cap L_r^\lambda) \mathrm{d}L_r^\lambda =$$

$$\sum_{j=0}^{(r-1)/2}\left(\sum_{i=j}^{(r-1)/2}\binom{r-1}{2i}\frac{2c^n_{i-j,2i,r}}{O_{2i}O_{r-2i-1}}(\lambda^2-1)^{\frac{r-2i-1}{2}}\lambda^{2i-2j}\right)M_{2j}(\partial Q).$$

Proof: For every L_r^λ intersecting Q, the Gauss–Bonnet formula in space forms of sectional curvature (λ^2-1) states (cf. [San76]), for r even,

$$\frac{O_r}{2}\chi(Q \cap L_r^\lambda) = (\lambda^2-1)^{r/2} V(Q \cap L_r^\lambda)$$

$$+\sum_{i=1}^{r/2}\binom{r-1}{2i-1}\frac{O_r}{O_{2i-1}O_{r-2i}}(\lambda^2-1)^{(r-2i)/2}M_{2i-1}(\partial Q \cap L_r^\lambda);$$

and for r odd,

$$\frac{O_r}{2}\chi(Q \cap L_r^\lambda) = \sum_{i=0}^{(r-1)/2}\binom{r-1}{2i}\frac{O_r}{O_{2i}O_{r-2i-1}}(\lambda^2-1)^{(r-2i-1)/2}M_{2i}(\partial Q \cap L_r^\lambda).$$

Integrating with respect to L_r^λ, in the even case

$$\frac{O_r}{2}\int_{\mathcal{L}_r^\lambda} \chi(Q \cap L_r^\lambda) \mathrm{d}L_r^\lambda = (\lambda^2-1)^{r/2}\int_{\mathcal{L}_r^\lambda} V(Q \cap L_r^\lambda)\mathrm{d}L_r^\lambda$$

$$+\sum_{i=1}^{r/2}\binom{r-1}{2i-1}\frac{O_r}{O_{2i-1}O_{r-2i}}(\lambda^2-1)^{(r-2i)/2}\int_{\mathcal{L}_r^\lambda} M_{2i-1}(\partial Q \cap L_r^\lambda)\mathrm{d}L_r^\lambda,$$

and by Corollary 4.1 and Proposition 3.2,

$$\frac{O_r}{2}\int_{\mathcal{L}_r^\lambda}\chi(Q\cap L_r^\lambda)\mathrm{d}L_r^\lambda=(\lambda^2-1)^{r/2}\frac{O_{n-1}\cdots O_{n-r-1}}{O_{r-1}\cdots O_0}\cdot V(Q)$$

$$+\sum_{i=1}^{r/2}\binom{r-1}{2i-1}\frac{O_r}{O_{2i-1}O_{r-2i}}(\lambda^2-1)^{(r-2i)/2}\Big(\sum_{l=0}^{i-1}c_{l,2i-1,r}^n\lambda^{2l}M_{2i-2l-1}(\partial Q)\Big)$$

and reordering the sums we get the sought formula. In the odd case one proceeds analogously. ∎

For $\lambda=1$ and $r=n-1$ we get the integral of the Euler characteristic of intersections with horospheres (as in [San67], [San68] and [GNS]).

Remark: These results are of special interest in the case of λ-convex domains (cf. [GaRe]). A domain Q is called *λ-convex* if for every λ-geodesic curve L_1^λ, the intersection $L_1^\lambda\cap Q$ is connected. In this case $L_r^\lambda\cap Q$ is contractible for any L_r^λ, so the last formulas give the measure of λ-geodesic r-planes that intersect Q. For instance, the measure of λ-planes in $\mathbb{H}^3$ intersecting a λ-convex domain is

$$\int_{L_2^\lambda\cap Q\neq\emptyset}\mathrm{d}L_2^\lambda=2M_1(\partial Q)-(1-\lambda^2)V(Q).$$

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